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Generalized Lie derivations on triangular algebras

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ABSTRACT

Let \mathcal{A} be a unital algebra and let \mathcal{M} be a unitary \mathcal{A} -bimodule. We consider generalized Lie derivations mapping from \mathcal{A} to \mathcal{M} . Our results are applied to triangular algebras, in particular to nest algebras and (block) upper triangular matrix algebras. We prove that under certain conditions each generalized Lie derivation of a triangular algebra \mathcal{A} is the sum of a generalized derivation and a central map which vanishes on all commutators of \mathcal{A} .

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1. Introduction

Let R be a commutative ring with identity. Let \mathcal{A} be an algebra over R and \mathcal{M} be a unitary \mathcal{A} -bimodule. By $[x, y] = xy - yx$ we denote the commutator or the Lie product of elements $x, y \in \mathcal{A}$, respectively. Similarly, we use $[m, x]$ to denote $mx - xm$ for any $x \in \mathcal{A}$ and for any $m \in \mathcal{M}$. A linear map $f : \mathcal{A} \rightarrow \mathcal{M}$ is a *generalized Lie derivation* if there exists a linear map $d : \mathcal{A} \rightarrow \mathcal{M}$ such that

$$f([x, y]) = f(x)y - f(y)x + xd(y) - yd(x) \quad \text{for all } x, y \in \mathcal{A}.$$

Any Lie derivation $f : \mathcal{A} \rightarrow \mathcal{M}$, a linear map which satisfies $f([x, y]) = [f(x), y] + [x, f(y)]$ for all $x, y \in \mathcal{A}$, is a generalized Lie derivation for $d = f$. We will call a linear map $f : \mathcal{A} \rightarrow \mathcal{M}$ a *generalized derivation* if there exists a linear map $d : \mathcal{A} \rightarrow \mathcal{M}$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in \mathcal{A}$. Any generalized derivation is a generalized Lie derivation.

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Let $Z(\mathcal{M}) = \{m \in \mathcal{M}; [m, \mathcal{A}] = 0\}$ be the center of \mathcal{M} . Note that every linear map $\tau : \mathcal{A} \rightarrow Z(\mathcal{M})$ that vanishes on all commutators of \mathcal{A} is a Lie derivation. A standard example of a Lie derivation is therefore a map of the form $f = d + \tau$, where $d : \mathcal{A} \rightarrow \mathcal{M}$ is a derivation and $\tau : \mathcal{A} \rightarrow Z(\mathcal{M})$ is a linear map such that $\tau([\mathcal{A}, \mathcal{A}]) = 0$. The natural problem that one considers in this context is whether every Lie derivation on a given algebra is of the standard form. We refer the reader to [1,2,9,13,23,26,27,31] for more details about the history and the importance of this problem.

The concept of a generalized derivation was introduced by Brešar [7] and generalized by Hvala [21], who has proved in [22] that each generalized Lie derivation of a prime ring is the sum of a generalized derivation and a central map which vanishes on all commutators. Recently, Liao and Liu [25] have generalized this result, describing all generalized Lie derivations on a Lie ideal of a prime algebra. In all these papers the theory of functional identities has been applied (more on functional identities and their applications can be found in book [11]). One of the reasons for studying generalized derivations on triangular algebras are the following recently published papers. Hou and Qi [19,20] described generalized Jordan derivations and generalized Lie derivations on nest algebras under the additional assumption that the associated map d is a Jordan derivation or a Lie derivation, respectively. Further, Ma and Ji [28] described generalized Jordan derivations mapping from a triangular matrix algebra $\mathcal{T}_n(R)$ into a $\mathcal{T}_n(R)$ -bimodule. Upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras are basic examples of triangular algebras. Study of different maps on triangular algebras is an active research area (see [5,6,13–15,17,30,33,34,36]). Since a triangular algebra has an identity, we will work in general with assumption that \mathcal{A} is an unital algebra and that considered maps map into \mathcal{A} -bimodules.

The paper is organized as follows: in Section 2 we shortly mention some of the properties and examples of a triangular algebras. Section 3 discusses generalized Lie derivations. As already mentioned earlier, was this question studied in [20] with additional condition that related map d is a Lie derivation. If we assume only that $d : \mathcal{A} \rightarrow \mathcal{M}$ is a linear map then the problem is more interesting and unpredictable. In general it does not follows that the related map d is necessary a Lie derivation (see Example 1). We prove in Lemma 3.2 that an element $m = d(1) \in \mathcal{M}$ satisfies special identity

$$[x, y][z, m] + [y, z][x, m] + [z, x][y, m] = 0 \quad \text{for all } x, y, z \in \mathcal{A}.$$

If every element $m \in \mathcal{M}$ which solves desired identity has the so called standard form ($m = m_1 + m_2$ where $m_1 \in Z(\mathcal{M})$ and $[\mathcal{A}, \mathcal{A}]m_2 = 0$), then the Theorem 3.3 states that each generalized Lie derivation $f : \mathcal{A} \rightarrow \mathcal{M}$ is the sum of a generalized derivation $\Delta : \mathcal{A} \rightarrow \mathcal{M}$ and a Lie derivation $\delta : \mathcal{A} \rightarrow \mathcal{M}$. Examples 2 and 3 show that if there exist elements of the bimodule \mathcal{M} which satisfy desired identity and do not have standard form, then there exist generalized Lie derivations which are not the sums of the generalized derivations and Lie derivations. Elements of triangular algebras which satisfies desired identity have in the certain conditions standard form (Lemma 3.5). The section is concluded with Theorem 3.6, that states that if triangular algebra \mathcal{A} satisfies properties of center projections, then each generalized Lie derivation $f : \mathcal{A} \rightarrow \mathcal{A}$ is the sum of generalized derivation $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ and linear map $\tau : \mathcal{A} \rightarrow Z(\mathcal{A})$ which maps commutator $[\mathcal{A}, \mathcal{A}]$ to 0.

In last section we study generalized Jordan derivations on unital algebras. The basic result Lemma 4.1 states that each generalized Jordan derivation which maps from \mathcal{A} into an unital \mathcal{A} -bimodule \mathcal{M} is the sum of a generalized derivation and a Jordan derivation. Our method enables us to give very short proofs of the main results in [19,28].

2. Preliminaries

Let A and B be unital algebras over a commutative ring R , and let M be a unital (A, B) -bimodule, which is faithful as a left A -module and also as a right B -module. Recall that a left A -module M is faithful if $aM = 0$ implies that $a = 0$. The R -algebra

$$\mathcal{A} = \text{Tri}(A, M, B) = \left\{ \begin{pmatrix} a & m \\ & b \end{pmatrix}; a \in A, m \in M, b \in B \right\}$$

under the usual matrix operations is called a triangular algebra. Let 1_A and 1_B be identities of algebras A and B , respectively, and let 1 be the identity of the triangular algebra \mathcal{A} . Throughout this paper we shall use the following notation

$$e = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}, \quad f = 1 - e = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}.$$

We immediately notice that e and f are orthogonal idempotents of \mathcal{A} and so \mathcal{A} may be represented as

$$\mathcal{A} = e\mathcal{A}e + e\mathcal{A}f + f\mathcal{A}f.$$

Here $e\mathcal{A}e$ is a subalgebra of \mathcal{A} isomorphic to A , $f\mathcal{A}f$ is a subalgebra of \mathcal{A} isomorphic to B and $e\mathcal{A}f$ is a $(e\mathcal{A}e, f\mathcal{A}f)$ -bimodule isomorphic to bimodule M . In order to simplify the notation we will use the following convention: $a = eae \in A = e\mathcal{A}e$, $b = fbf \in B = f\mathcal{A}f$ and $m = emf \in M = e\mathcal{A}f$. Then each element $x \in \mathcal{A}$ can be represented in the form $x = eae + emf + fbf = a + m + b$, where $a \in A$, $b \in B$, $m \in M$. Let us define two natural projections $\pi_A : \mathcal{A} \rightarrow A$ and $\pi_B : \mathcal{A} \rightarrow B$ by

$$\pi_A(a + m + b) = a \quad \text{and} \quad \pi_B(a + m + b) = b.$$

By [13, Proposition 3] we know that $Z(\mathcal{A}) = \{a + b \mid am = mb \text{ for all } m \in M\}$. Moreover, $\pi_A(Z(\mathcal{A})) \subseteq Z(A)$ and $\pi_B(Z(\mathcal{A})) \subseteq Z(B)$, and there exists a unique algebra isomorphism $\psi : \pi_A(Z(\mathcal{A})) \rightarrow \pi_B(Z(\mathcal{A}))$ such that $am = m\psi(a)$ for all $m \in M$. The most important examples of triangular algebras are upper triangular matrix algebras, block upper triangular matrix algebras and nest algebras over a real or a complex Banach space X or Hilbert space H , respectively.

Upper triangular matrix algebras. Let $\mathcal{M}_{k \times m}(R)$ be the set of all $k \times m$ matrices and let $\mathcal{T}_n(R)$ be the algebra of all $n \times n$ upper triangular matrices over R . For $n \geq 2$ and each $1 \leq l \leq n-1$ the algebra $\mathcal{T}_n(R)$ can be represented as a triangular algebra of the form $\mathcal{T}_n(R) = \text{Tri}(\mathcal{T}_l(R)_l, \mathcal{M}_{l \times (n-l)}(R), \mathcal{T}_{n-l}(R))$. Since $\mathcal{T}_n(R)$ is a central algebra it follows $\pi_A(Z(\mathcal{A})) = Z(A)$ and $\pi_B(Z(\mathcal{A})) = Z(B)$.

Block upper triangular matrix algebras. Let \mathbb{N} be the set of all positive integers and let $n \in \mathbb{N}$. For every positive integer m , $m \leq n$, we denote by $\bar{k} = (k_1, k_2, \dots, k_m) \in \mathbb{N}^m$ an ordered m -vector of positive integers such that $n = k_1 + k_2 + \dots + k_m$. The block upper triangular matrix algebra $\mathcal{B}_n(R)$ with corresponding vector \bar{k} is a subalgebra of $\mathcal{M}_n(R)$ which contains all block upper triangular matrices, where diagonal blocks have sizes k_1, k_2, \dots, k_m . Note that $\mathcal{M}_n(R)$ and $\mathcal{T}_n(R)$ are two special cases of block upper triangular matrix algebras. If we have $n \geq 2$ and $\mathcal{B}_n(R) \neq \mathcal{M}_n(R)$ then $\mathcal{B}_n(R)$ is a triangular algebra and can be represented as $\mathcal{B}_n(R) = \text{Tri}(\mathcal{B}_l(R), \mathcal{M}_{l \times (n-l)}(R), \mathcal{B}_{n-l}(R))$ where $1 \leq l < m$ and $\mathcal{B}_l(R)$, $\mathcal{B}_{n-l}(R)$ are block upper triangular matrix algebras with suitable vectors $\bar{k}_1 \in \mathbb{N}^l$, $\bar{k}_2 \in \mathbb{N}^{m-l}$. Since $\mathcal{B}_n(R)$ is a central algebra it follows $\pi_A(Z(\mathcal{A})) = Z(A)$ and $\pi_B(Z(\mathcal{A})) = Z(B)$.

Nest algebras. A nest is a chain \mathcal{N} of closed subspaces of a real or complex Hilbert space H (Banach space X) containing $\{0\}$ and H which is closed under arbitrary intersections and closed linear span. The nest algebra associated to \mathcal{N} is the algebra

$$\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(H) \mid T(N) \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

A nest \mathcal{N} is called trivial if $\mathcal{N} = \{0, H\}$. A nontrivial nest algebra over a Hilbert space H is a triangular algebra. Namely, if $N \in \mathcal{N} \setminus \{0, H\}$ and E is the orthonormal projection onto N , then $\mathcal{N}_1 = E(\mathcal{N})$ and $\mathcal{N}_2 = (1 - E)(\mathcal{N})$ are nests of N and N^\perp , respectively. Moreover, $\mathcal{T}(\mathcal{N}_1) = E\mathcal{T}(\mathcal{N})E$, $\mathcal{T}(\mathcal{N}_2) = (1 - E)\mathcal{T}(\mathcal{N})(1 - E)$ are nest algebras and $\mathcal{T}(\mathcal{N}) = \text{Tri}(\mathcal{T}(\mathcal{N}_1), E\mathcal{T}(\mathcal{N})(1 - E), \mathcal{T}(\mathcal{N}_2))$. Let us mention that finite dimensional nest algebras are isomorphic to a real or a complex block upper triangular matrix algebras. For a nontrivial nest \mathcal{N} we have again $\pi_A(Z(\mathcal{A})) = Z(A)$ and $\pi_B(Z(\mathcal{A})) = Z(B)$, where $\mathcal{A} = \mathcal{T}(\mathcal{N})$, $A = \mathcal{T}(\mathcal{N}_1)$, $B = \mathcal{T}(\mathcal{N}_2)$.

In case \mathcal{N} is a nest over a Banach space X then $\mathcal{T}(\mathcal{N})$ is a triangular algebra if there exists a subspace $N \in \mathcal{N} \setminus \{0, X\}$ which is complemented in X (for details of this construction see [20, proof of Theorem 2.2]).

3. Generalized Lie derivations

Let \mathcal{A} be an algebra with identity and let \mathcal{M} be a unital \mathcal{A} -bimodule. Recall us that a linear map $f : \mathcal{A} \rightarrow \mathcal{M}$ is generalized derivation, if there exists linear map $d : \mathcal{A} \rightarrow \mathcal{M}$, such that

$$f(xy) = f(x)y + xd(y) \quad \text{for all } x, y \in \mathcal{A}. \quad (3.1)$$

Let us begin with a description of generalized derivations:

Remark 3.1. Let \mathcal{A} be an algebra with identity and let \mathcal{M} be a unital \mathcal{A} -bimodule. Let $f : \mathcal{A} \rightarrow \mathcal{M}$ be a generalized derivation with an associated linear map d . Then d is a derivation and it follows that $f(x) = f(1)x + d(x)$ for all $x \in \mathcal{A}$.

Proof. Setting $x = 1$ in (3.1) we get $f(y) = f(1)y + d(y)$ for all $y \in \mathcal{A}$. Consequently, (3.1) implies

$$f(1)xy + d(xy) = f(1)xy + d(x)y + xd(y)$$

for all $x, y \in \mathcal{A}$. Therefore, d is a derivation. \square

We ask ourselves, when is a generalized derivation $f : \mathcal{A} \rightarrow \mathcal{M}$ an *inner generalized derivation*, i.e. a map of the form $f(x) = mx + xn$, where $m, n \in \mathcal{M}$. This is true iff d is an inner derivation of the form $d(x) = [m, x]$, where $m \in \mathcal{M}$. The question of innerness of derivations on triangular algebras was studied in [5, 14, 17].

Let us recall that a linear map $f : \mathcal{A} \rightarrow \mathcal{M}$ is a generalized Lie derivation, if there exists a linear map $d : \mathcal{A} \rightarrow \mathcal{M}$, such that

$$f([x, y]) = f(x)y - f(y)x + xd(y) - yd(x) \quad \text{for all } x, y \in \mathcal{A}. \quad (3.2)$$

If we set $y = 1$ in (3.2) we get

$$f(x) = f(1)x - xd(1) + d(x) \quad \text{for all } x \in \mathcal{A}. \quad (3.3)$$

Using (3.3) and (3.2) we obtain

$$d([x, y]) = [d(x), y] + [x, d(y)] + x[y, d(1)] + y[d(1), x] \quad (3.4)$$

for all $x, y \in \mathcal{A}$. We see that a linear map d is a Lie derivation iff $d(1) \in Z(\mathcal{M})$. Hence, if $d(1) \in Z(\mathcal{M})$ it follows from (3.3) that a generalized Lie derivation f is the sum of a generalized inner derivation and a Lie derivation. Let us define a map $\delta : \mathcal{A} \rightarrow \mathcal{M}$ as

$$\delta(x) = d(x) - xd(1) \quad \text{for all } x \in \mathcal{A}.$$

If we use $d(x) = \delta(x) + xd(1)$ in identity (3.4) it follows

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)] + [x, y]d(1) \quad \text{for all } x, y \in \mathcal{A}. \quad (3.5)$$

We see, that δ is a Lie derivation iff $[\mathcal{A}, \mathcal{A}]d(1) = 0$. Thus, $f(x) = f(1)x + \delta(x)$ for all $x \in \mathcal{A}$ and so f is again the sum of a generalized inner derivation and a Lie derivation if $[\mathcal{A}, \mathcal{A}]d(1) = 0$. The form of a generalized Lie derivation is determined by the image of the identity. Before we study properties of the element $d(1)$, we mention some natural examples of maps satisfying (3.4).

Example 1. Each derivation $d : \mathcal{A} \rightarrow \mathcal{M}$ satisfies (3.4) since $d(1) = 0$. Let $\tau : \mathcal{A} \rightarrow Z(\mathcal{M})$ be a linear map such that $\tau([\mathcal{A}, \mathcal{A}]) = 0$. Since $\tau(1) \in Z(\mathcal{M})$, we see that τ satisfies (3.4). Thus, each standard

Lie derivation is a solution of identity (3.4). Let $m \in \mathcal{M}$ be an element such that $[\mathcal{A}, \mathcal{A}]m = 0$. We define $d(x) = mx$ for all $x \in \mathcal{A}$. We consider $d(1) = m$ and use map d in (3.4) we obtain

$$\begin{aligned} & [mx, y] + [x, my] + x[y, m] + y[m, x] \\ &= mxy - ymx + xmy - myx + xym - xmy + ymx - yxm \\ &= m[x, y] + [x, y]m \\ &= m[x, y] \end{aligned}$$

for all $x, y \in \mathcal{A}$. Therefore $d(x) = mx$ satisfies equality (3.4). Similarly one can prove, that the right multiplication with the element $m \in \mathcal{M}$, $d(x) = xm$ for all $x \in \mathcal{A}$, satisfies (3.4). A natural candidate for the map satisfying (3.4) is also a generalized inner derivation $\Delta(x) = mx + xn$, where elements $m, n \in \mathcal{M}$ satisfy $[\mathcal{A}, \mathcal{A}]m = [\mathcal{A}, \mathcal{A}]n = 0$.

Lemma 3.2. *Let $f : \mathcal{A} \rightarrow \mathcal{M}$ be a generalized Lie derivation with a related linear map d . Then*

$$[x, y][z, d(1)] + [y, z][x, d(1)] + [z, x][y, d(1)] = 0 \quad (3.6)$$

for all $x, y, z \in \mathcal{A}$.

Proof. In this proof we use the well known Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. Recall that $\delta(x) = d(x) - xd(1)$ for all $x \in \mathcal{A}$. Using identity (3.5) we consider the action of δ on the triple Lie product $[[x, y], z]$:

$$\begin{aligned} \delta([[x, y], z]) &= [\delta([x, y]), z] + [[x, y], \delta(z)] + [[x, y], z]d(1) \\ &= [[\delta(x), y], z] + [[x, \delta(y)], z] + [[x, y], \delta(z)] \\ &\quad + [[x, y]d(1), z] + [[x, y], z]d(1) \end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Similarly we can write

$$\begin{aligned} \delta([[y, z], x]) &= [[\delta(y), z], x] + [[y, \delta(z)], x] + [[y, z], \delta(x)] \\ &\quad + [[y, z]d(1), x] + [[y, z], x]d(1) \end{aligned}$$

and

$$\begin{aligned} \delta([[z, x], y]) &= [[\delta(z), x], y] + [[z, \delta(x)], y] + [[z, x], \delta(y)] \\ &\quad + [[z, x]d(1), y] + [[z, x], y]d(1) \end{aligned}$$

for all $x, y, z \in \mathcal{A}$. Summing up all three equalities and using the Jacobi identity we obtain

$$0 = [[x, y]d(1), z] + [[y, z]d(1), x] + [[z, x]d(1), y] \quad (3.7)$$

for all $x, y, z \in \mathcal{A}$. Now, we rewrite identity (3.7) as

$$\begin{aligned} 0 &= [x, y][d(1), z] + [y, z][d(1), x] + [z, x][d(1), y] \\ &\quad + ([x, y], z) + ([y, z], x) + ([z, x], y)d(1). \end{aligned}$$

Applying the Jacobi identity once again, we obtain the desired result. \square

The question arises, which elements $m \in \mathcal{M}$ satisfy identity

$$[x, y][z, m] + [y, z][x, m] + [z, x][y, m] = 0 \quad \text{for all } x, y, z \in \mathcal{A}. \quad (3.8)$$

Clearly each $m \in Z(\mathcal{M})$ solves this identity. Also, if $[\mathcal{A}, \mathcal{A}]m = 0$ then identity (3.8) is true. Namely, from

$$[x, yz]m = y[x, z]m + [x, y]zm \quad \text{for all } x, y, z \in \mathcal{A}$$

it follows $[x, y]zm = 0$ for all $x, y, z \in \mathcal{A}$. Therefore $[x, y][z, m] = 0$ for all $x, y, z \in \mathcal{A}$ and identity (3.8) is true. If $m = m_1 + m_2$, where $m_1 \in Z(\mathcal{M})$ and $[\mathcal{A}, \mathcal{A}]m_2 = 0$, then m is also the solution of the identity (3.8). Our desire is that each element $d(1) = m$ which satisfies identity (3.8) is of this form, since then follows:

Theorem 3.3. *Let \mathcal{A} be an algebra with identity and let \mathcal{M} be a unital \mathcal{A} -bimodule. Let $f : \mathcal{A} \rightarrow \mathcal{M}$ be a generalized Lie derivation with an associated linear map d . Suppose that for each $m \in \mathcal{M}$ satisfying*

$$[x, y][z, m] + [y, z][x, m] + [z, x][y, m] = 0 \quad \text{for all } x, y, z \in \mathcal{A}.$$

there exist $m_1, m_2 \in \mathcal{M}$ such that $m = m_1 + m_2$, $m_1 \in Z(\mathcal{M})$, and $[\mathcal{A}, \mathcal{A}]m_2 = 0$. Then $f = \Delta + \delta$, where $\Delta : \mathcal{A} \rightarrow \mathcal{M}$ is a generalized inner derivation and $\delta : \mathcal{A} \rightarrow \mathcal{M}$ is a Lie derivation.

Proof. Let $f : \mathcal{A} \rightarrow \mathcal{M}$ be a generalized Lie derivation with corresponding linear map d . From Lemma 3.2 it follows that element $m = d(1)$ satisfies given identity. Thus, according to our assumption $m = m_1 + m_2$ for some $m_1, m_2 \in \mathcal{M}$ such that $m_1 \in Z(\mathcal{M})$ and $[\mathcal{A}, \mathcal{A}]m_2 = 0$. Let us define $\delta(x) = d(x) - xm_2$ and $\Delta(x) = f(1)x - xm_1$ for all $x \in \mathcal{A}$. Obviously, Δ is a generalized inner derivation. Now, using (3.3) we obtain

$$\begin{aligned} f(x) &= f(1)x - xd(1) + d(x) \\ &= f(1)x - xm_1 + d(x) - xm_2 \\ &= \Delta(x) + \delta(x) \end{aligned}$$

for all $x \in \mathcal{A}$. Since $d(x) = \delta(x) + xm_2$ and $[\mathcal{A}, \mathcal{A}]m_2 = 0$ identity (3.4) implies

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)] \quad \text{for all } x, y \in \mathcal{A}.$$

Thus, f is the sum of a generalized inner derivation and a Lie derivation. \square

Let $f : \mathcal{A} \rightarrow \mathcal{M}$ be a generalized Lie derivation with an associated linear map d . When the assumptions of Theorem 3.3 are fulfilled then d is also the sum of a generalized inner derivation and a Lie derivation. Namely, from (3.3) we see that the difference $f - d$ is a generalized inner derivation. As an application of Theorem 3.3 we obtain the following result.

Corollary 3.4. *Let $\mathcal{M}_n(R)$ be a matrix algebra over R and let \mathcal{M} be a unital 2-torsionfree $\mathcal{M}_n(R)$ -bimodule. Let $f : \mathcal{M}_n(R) \rightarrow \mathcal{M}$ be a generalized Lie derivation with an associated linear map d . Then $f = \Delta + \tau$, where $\Delta : \mathcal{M}_n(R) \rightarrow \mathcal{M}$ is a generalized derivation and $\tau : \mathcal{M}_n(R) \rightarrow Z(\mathcal{M})$ is a linear map that vanishes on all commutators of $\mathcal{M}_n(R)$.*

Proof. Let $\{e_{ij} | i, j = 1, 2, \dots, n\}$ be the system of matrix units of $\mathcal{M}_n(R)$ and let $1 = e_{11} + e_{22} + \dots + e_{nn}$ be the identity of algebra $\mathcal{M}_n(R)$. Since \mathcal{M} is a unital $\mathcal{M}_n(R)$ -bimodule each $m \in \mathcal{M}$ can be written in the form

$$m = 1m1 = \sum_{i,j=1}^n e_{ii}me_{jj} = \sum_{i,j=1}^n m_{ij},$$

where $m_{ij} = e_{ii}me_{jj}$ for all i, j . Note that

$$Z(\mathcal{M}) = \{m = m_{11} + m_{22} + \dots + m_{nn} | m_{ij} = e_{ji}m_{ii}e_{ij} \text{ for all } i, j\}.$$

We prove that the central elements of \mathcal{M} are the only ones that satisfy identity (3.8). Suppose that $m \in \mathcal{M}$ satisfies (3.8). Let i and j be arbitrary indices such that $i \neq j$. We fix matrix units $x = e_{ii}, y = e_{ij}$ and $z = e_{ji}$ and use them in (3.8). We get the equality

$$e_{ij}[e_{ji}, m] + (e_{ii} - e_{jj})[e_{ii}, m] + e_{ji}[e_{ij}, m] = 0.$$

According to $m = \sum_{i,j=1}^n m_{ij}$ we can write

$$e_{ij} \sum_{k=1}^n (e_{ji} m_{ik} - m_{kj} e_{ji}) + (e_{ii} - e_{jj}) \sum_{k=1}^n (m_{ik} - m_{ki}) + e_{ji} \sum_{k=1}^n (e_{ij} m_{jk} - m_{ki} e_{ij}) = 0.$$

Then it follows

$$2 \sum_{\substack{k=1 \\ k \neq i}}^n m_{ik} + \sum_{\substack{k=1 \\ k \neq j, k \neq i}}^n m_{jk} + (m_{ii} - e_{ij} m_{jj} e_{ji}) + 2m_{ji} + (m_{jj} - e_{ji} m_{ii} e_{ij}) = 0.$$

If we multiply given identity from left and right side with suitable matrix units e_{ii} and since \mathcal{M} is 2-torsionfree bimodule, it follows $m_{ik} = 0$ for all $k \neq i$, $m_{jk} = 0$ for all $k \neq j$ and $m_{jj} = e_{ji} m_{ii} e_{ij}$. Since i and j were arbitrary indices then $m = \sum_{k=1}^n m_{kk}$ and $m_{jj} = e_{ji} m_{ii} e_{ij}$ for all i, j . Therefore $m \in Z(\mathcal{M})$.

Let $f : \mathcal{M}_n(R) \rightarrow \mathcal{M}$ be a generalized Lie derivation with an associated linear map d . From Theorem 3.3 it follows that f is the sum of a generalized derivation $\Delta_1 : \mathcal{M}_n(R) \rightarrow \mathcal{M}$ and a Lie derivation $\delta : \mathcal{M}_n(R) \rightarrow \mathcal{M}$. Now, using the same arguments as in the proof of Theorem [23, Theorem 9.4] we conclude that each Lie derivation $\delta : \mathcal{M}_n(R) \rightarrow \mathcal{M}$ is the sum of a derivation Δ_2 and a central linear map τ that vanishes on all commutators of $\mathcal{M}_n(R)$. Finally, we set $\Delta = \Delta_1 + \Delta_2$, then Δ is a generalized derivation and $f = \Delta + \tau$. Therefore the proof is complete. \square

In general there exist generalized Lie derivations, which are not the sums of generalized derivation and a Lie derivation. We will present two such examples. We can ask ourselves, if Proposition 3.4 holds true also for upper triangular matrix algebra $\mathcal{T}_n(R)$? Namely, in [4] it is proved, that each Lie derivation from algebra $\mathcal{T}_n(R)$ into its bimodule is of the standard form. Unfortunately, there exist generalized Lie derivations on $\mathcal{T}_n(R)$ which are not the sums of a generalized derivation and a Lie derivation. Such a generalized Lie derivation $f : \mathcal{T}_2(R) \rightarrow \mathcal{M}_2(R)$ is constructed in Example 2. In this section we will also study generalized Lie derivations which map from triangular algebra $\text{Tri}(A, M, B)$ into itself. Therefore, a generalized Lie derivation of a triangular algebra, which is not the sum of a generalized derivation and a Lie derivation is constructed in Example 3.

Example 2. Let $\mathcal{A} = \mathcal{T}_2(R)$ and let $\mathcal{M} = \mathcal{M}_2(R)$, which is a unital $\mathcal{T}_2(R)$ -bimodule. The commutator of algebra \mathcal{A} is generated by matrix unit e_{12} , i.e. $[\mathcal{A}, \mathcal{A}] = \{re_{12} | r \in R\}$. By direct computation one can verify that each $m \in \mathcal{M}$ satisfies identity (3.8). However, element

$$m_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{M}_2(R)$$

is not of the form $m_0 = m_1 + m_2$, where $m_1 \in Z(\mathcal{M})$ and $[\mathcal{A}, \mathcal{A}]m_2 = 0$. Namely, elements m_1 and m_2 are of the form:

$$m_1 = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \quad \text{and} \quad m_2 = \begin{pmatrix} t & s \\ 0 & 0 \end{pmatrix},$$

where $r, t, s \in R$. Thus, in this case the assumptions of Theorem 3.3 are not fulfilled.

We use element $m_0 = e_{21}$ for the construction of a generalized Lie derivation which is not the sum of a generalized derivation and a Lie derivation. We define linear maps $f, d : \mathcal{A} \rightarrow \mathcal{M}$ by

$$f \begin{pmatrix} r & t \\ s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ r-s & t \end{pmatrix} \quad \text{and} \quad d \begin{pmatrix} r & t \\ s \end{pmatrix} = \begin{pmatrix} t & 0 \\ r & t \end{pmatrix}$$

for all $r, t, s \in R$. Note that $d(1) = e_{21}$ and $f(x) = d(x) - xd(1)$ for all $x \in \mathcal{A}$. By straightforward computation one can prove, that f is a generalized Lie derivation with associated linear map d . Since $f(1) = 0$ we can according to (3.3) and (3.5) write

$$f([x, y]) = [f(x), y] + [x, f(y)] + [x, y]d(1) \quad \text{for all } x, y \in \mathcal{A}. \quad (3.9)$$

Now assume, that f is the sum of a generalized derivation and a Lie derivation. Each generalized derivation $\Delta : \mathcal{A} \rightarrow \mathcal{M}$ is of the form $\Delta(x) = \Delta(1)x + d'(x)$ for all $x \in \mathcal{A}$, where d' is a derivation. Since each derivation is also a Lie derivation, we can assume that f is of the form $f(x) = mx + \delta(x)$ for all $x \in \mathcal{A}$, where $m \in \mathcal{M}$ and δ is a Lie derivation. Consequently, (3.9) implies

$$0 = x(my + yd(1)) - y(mx + xd(1)) \quad \text{for all } x, y \in \mathcal{A}. \quad (3.10)$$

If we set $y = 1$, we see that $m \in Z(\mathcal{M})$. Let us denote $m = \lambda e_{11} + \lambda e_{22}$ for some $\lambda \in R$. Hence the equality (3.10) can be rewritten into $[x, y](m + d(1)) = 0$ for all $x, y \in \mathcal{A}$. Finally, we set $x = e_{11}$, $y = e_{12}$ and obtain

$$[x, y](m + d(1)) = e_{12}(\lambda e_{11} + \lambda e_{22} + e_{21}) = e_{11} + \lambda e_{12} \neq 0.$$

This is a contradiction. Thus f is not the sum of a generalized derivation and a Lie derivation.

Example 3. Suppose that $R[X]$ is a polynomial ring with coefficients from commutative ring R with identity. Let $A = R$ and let $B = R[X]/(X^2)$ be a quotient R -algebra. Then $M = B$ is faithful as left A -module and also as a right B -module. We construct a triangular algebra $\mathcal{A} = \text{Tri}(A, M, B)$ of the form

$$\mathcal{A} = \left\{ \begin{pmatrix} r & t_0 + t_1 X \\ s_0 + s_1 X \end{pmatrix}; r, t_0, t_1, s_0, s_1 \in R \right\}.$$

Note that $Z(\mathcal{A}) = R1$. Since both algebras A and B are commutative it follows that $[\mathcal{A}, \mathcal{A}] = e\mathcal{A}f$. Hence $[\mathcal{A}, \mathcal{A}][\mathcal{A}, \mathcal{A}] = 0$ and so $[x, y][z, x_0] + [y, z][x, x_0] + [z, x][y, x_0] = 0$ for all $x, x_0, y, z \in \mathcal{A}$. Setting

$$x_0 = \begin{pmatrix} 0 & 0 \\ X \end{pmatrix} \in \mathcal{A}$$

we see that x_0 is not of the form $x_1 + x_2$, where $x_1 \in Z(\mathcal{A})$ and $[\mathcal{A}, \mathcal{A}]x_2 = 0$. Namely,

$$x_1 = \begin{pmatrix} r & 0 \\ r \end{pmatrix} \quad \text{and} \quad x_2 = \begin{pmatrix} s & t_0 + t_1 X \\ 0 \end{pmatrix},$$

for some $r, t_0, t_1, s \in R$.

Similarly to the previous example we use x_0 for the construction of a generalized Lie derivation, which is not the sum of a generalized derivation and a Lie derivation. Let us define a linear map $d : \mathcal{A} \rightarrow \mathcal{A}$ as

$$d \begin{pmatrix} r & t_0 + t_1 X \\ s_0 + s_1 X \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ rX \end{pmatrix}$$

for all $r, t_0, t_1, s_0, s_1 \in R$. Let $f(x) = d(x) - xd(1)$ for all $x \in \mathcal{A}$. By a straightforward computation one can prove, that f is a generalized Lie derivation with associated linear map d . Now, assume that f is the sum of a generalized derivation and a Lie derivation. Using the same arguments as in previous example we obtain $[x, y](x_1 + d(1)) = 0$ for all $x, y \in \mathcal{A}$, where $x_1 = \lambda 1 \in Z(\mathcal{A})$. Setting

$$x = \begin{pmatrix} 1 & 0 \\ 0 & \end{pmatrix} \quad \text{in} \quad y = \begin{pmatrix} 0 & 1 \\ 0 & \end{pmatrix}$$

in the last identity we get

$$[x, y](x_1 + d(1)) = \begin{pmatrix} 0 & \lambda - X \\ 0 & \end{pmatrix} \neq 0,$$

a contradiction. Thus, f is not the sum of a generalized derivation and a Lie derivation.

Note that in Example 3 we constructed a generalized Lie derivation on a triangular algebra $\mathcal{A} = \text{Tri}(A, M, B)$ such that $[A, A] = 0$ and $\pi_B(Z(\mathcal{A})) \neq Z(B)$. We have seen that this algebra has an element x_0 which satisfies identity (3.11), but is not of the standard form. This example is also the reason for the assumptions in the following lemma.

Lemma 3.5. *Let $\mathcal{A} = \text{Tri}(A, M, B)$ be a triangular algebra and let $x_0 \in \mathcal{A}$. Let us assume that*

$$[x, y][z, x_0] + [y, z][x, x_0] + [z, x][y, x_0] = 0 \quad \text{for all } x, y, z \in \mathcal{A}. \quad (3.11)$$

If an ideal of algebra A which is generated by commutators $[A, A]$ is equal to A or $\pi_B(Z(\mathcal{A})) = Z(B)$, then there exist $x_1, x_2 \in \mathcal{A}$ such that $x_0 = x_1 + x_2$, $x_1 \in Z(\mathcal{A})$, and $[\mathcal{A}, \mathcal{A}]x_2 = 0$.

Proof. Let $x_0 = a_0 + m_0 + b_0 \in \mathcal{A}$. Setting $x = e, y = m$ and $z = b$ in (3.11) we get $m[b, x_0] + mb[e, x_0] = 0$ for all $m \in M, b \in B$. We conclude $m[b, b_0] = 0$ for all $m \in M, b \in B$. Since M is faithful right B -module, we can conclude that $b_0 \in Z(B)$. Setting $x = a_1, y = a_2$ and $z = f$ in (3.11) we obtain $[a_1, a_2][f, x_0] = 0$ for all $a_1, a_2 \in A$. Consequently, $[a_1, a_2]m_0 = 0$ for all $a_1, a_2 \in A$ and hence

$$[A, A]m_0 = 0. \quad (3.12)$$

Setting $x = a_1, y = a_2$ and $z = m$ in (3.11) we obtain $[a_1, a_2][m, x_0] + a_2m[a_1, x_0] - a_1m[a_2, x_0] = 0$, which further implies $[a_1, a_2](mb_0 - a_0m) = 0$ for all $a_1, a_2 \in A, m \in M$. Therefore,

$$[A, A](mb_0 - a_0m) = 0 \quad \text{for all } m \in M. \quad (3.13)$$

Let J be an ideal of algebra A generated by $[A, A]$. It is easy to see that (3.12) implies $Jm_0 = 0$. Similarly, (3.13) yields $J(mb_0 - a_0m) = 0$ for all $m \in M$. If we assume that $J = A$ then $e \in J, m_0 = 0$, and $a_0m = mb_0$ for all $m \in M$. In this case $x_0 = a_0 + b_0 \in Z(\mathcal{A})$ is of the desired form.

Let us assume that $\pi_B(Z(\mathcal{A})) = Z(B)$. In this case $b_0 \in \pi_B(Z(\mathcal{A}))$. Let $x_1 = \psi^{-1}(b_0) + b_0 \in Z(\mathcal{A})$. We now rearrange (3.13) into

$$[A, A](a_0 - \psi^{-1}(b_0))m = 0 \quad \text{for all } m \in M.$$

Since M is faithful left A -module we get

$$[A, A](a_0 - \psi^{-1}(b_0)) = 0. \quad (3.14)$$

Let $x_2 = a_0 - \psi^{-1}(b_0) + m_0$. Since $[\mathcal{A}, \mathcal{A}] = [A, A] + M + [B, B]$ it follows from (3.12) and (3.14) that

$$[\mathcal{A}, \mathcal{A}]x_2 = [A, A]x_2 = 0.$$

Thus,

$$x_0 = a_0 - \psi^{-1}(b_0) + m_0 + \psi^{-1}(b_0) + b_0 = x_2 + x_1,$$

$[\mathcal{A}, \mathcal{A}]x_2 = 0$, and $x_1 \in Z(\mathcal{A})$. \square

Theorem 3.6. Let $\mathcal{A} = \text{Tri}(A, M, B)$ be a triangular algebra. Let $f : \mathcal{A} \rightarrow \mathcal{A}$ be a generalized Lie derivation with an associated linear map d . If the following conditions are met:

(i) $\pi_A(Z(\mathcal{A})) = Z(A)$ or $[B, B] = B$,

(ii) $\pi_B(Z(\mathcal{A})) = Z(B)$ or $[A, A] = A$,

then $f = \Delta + \tau$, where $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ is a generalized derivation and $\tau : \mathcal{A} \rightarrow Z(\mathcal{A})$ is a linear map that vanishes on all commutators of \mathcal{A} .

Proof. According to assumption (ii) the conditions of Lemma 3.5 are satisfied and hence Theorem 3.3 implies $f = \Delta_1 + \delta$, where $\Delta_1 : \mathcal{A} \rightarrow \mathcal{A}$ is a generalized inner derivation and $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is a Lie derivation. Using the result of Cheung [13, Theorem 11] we see that $\delta = \Delta_2 + \tau$, where $\Delta_2 : \mathcal{A} \rightarrow \mathcal{A}$ is a derivation and $\tau : \mathcal{A} \rightarrow Z(\mathcal{A})$ is a linear map vanishing on $[\mathcal{A}, \mathcal{A}]$. Denoting $\Delta = \Delta_1 + \Delta_2$ we conclude that $f = \Delta + \tau$. \square

Corollary 3.7. Let \mathcal{N} be a nest of a Banach space X and let $\mathcal{T}(\mathcal{N})$ be a nest algebra. Let $f : \mathcal{T}(\mathcal{N}) \rightarrow \mathcal{T}(\mathcal{N})$ be a generalized Lie derivation. If $\mathcal{N} = \{0, X\}$ or if there exists $N \in \mathcal{N} \setminus \{0, X\}$ which is complemented in X , then $f = \Delta + \tau$, where $\Delta : \mathcal{T}(\mathcal{N}) \rightarrow \mathcal{T}(\mathcal{N})$ is a generalized derivation and $\tau : \mathcal{T}(\mathcal{N}) \rightarrow \mathbb{F}1$ is a linear map that vanishes on all commutators of $\mathcal{T}(\mathcal{N})$.

Proof. If $\mathcal{N} = \{0, X\}$ is a trivial nest then $\mathcal{T}(\mathcal{N}) = \mathcal{B}(X)$. Since $\mathcal{B}(X)$ is a central prime algebra, we can apply [22] to obtain the conclusion the corollary. If there exists $N \in \mathcal{N} \setminus \{0, X\}$ which is complemented in X then $\mathcal{T}(\mathcal{N}) = \text{Tri}(A, M, B)$ is a triangular algebra and desired result follows from Theorem 3.6. Since all algebras $\mathcal{T}(\mathcal{N})$, A , B are central ($Z(\mathcal{T}(\mathcal{N})) = \mathbb{F}1$, $Z(A) = \mathbb{F}1_A$, $Z(B) = \mathbb{F}1_B$) we see that both conditions $\pi_A(Z(\mathcal{T}(\mathcal{N}))) = Z(A)$ and $\pi_B(Z(\mathcal{T}(\mathcal{N}))) = Z(B)$ are fulfilled. \square

Let H be a Hilbert space and let \mathcal{N} be a nest in a Hilbert space H . If $\mathcal{N} = \{0, X\}$ is trivial nest, then $\mathcal{T}(\mathcal{N}) = \mathcal{B}(H)$ is a prime algebra. If \mathcal{N} is a nontrivial nest, then $\mathcal{T}(\mathcal{N})$ is a triangular algebra. Since all derivations of $\mathcal{T}(\mathcal{N})$ are inner we obtain the following.

Corollary 3.8. Let \mathcal{N} be a nest of a Hilbert space H and let $\mathcal{T}(\mathcal{N})$ be a nest algebra. Each generalized Lie derivation $f : \mathcal{T}(\mathcal{N}) \rightarrow \mathcal{T}(\mathcal{N})$ is the sum of a generalized inner derivation $\Delta : \mathcal{T}(\mathcal{N}) \rightarrow \mathcal{T}(\mathcal{N})$ and a linear map $\tau : \mathcal{T}(\mathcal{N}) \rightarrow \mathbb{F}1$ that vanishes on $[\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N})]$.

Let $\mathcal{T}_n(R) \subseteq \mathcal{B}_n(R) \subset \mathcal{M}_n(R)$ be a proper block upper triangular matrix algebra over a commutative ring R .

Corollary 3.9. Each generalized Lie derivation $f : \mathcal{B}_n(R) \rightarrow \mathcal{B}_n(R)$ is the sum of a generalized inner derivation $\Delta : \mathcal{B}_n(R) \rightarrow \mathcal{B}_n(R)$ and a linear map $\tau : \mathcal{B}_n(R) \rightarrow R1$ that vanishes on all commutators of $\mathcal{B}_n(R)$.

Proof. According to [5, Proposition 3.6] all derivations of $\mathcal{B}_n(R)$ are inner and $\mathcal{B}_n(R)$ is a central algebra over $Z(\mathcal{B}_n(R)) = R1$. Since $\mathcal{B}_n(R) \neq \mathcal{M}_n(R)$ it follows that $\mathcal{B}_n(R)$ is a triangular algebra and the result follows directly from Theorem 3.6. \square

4. Generalized Jordan derivations

Let \mathcal{A} be an algebra with identity and let \mathcal{M} be a unital \mathcal{A} -bimodule. By $x \circ y = xy + yx$ we denote the Jordan product of elements $x, y \in \mathcal{A}$. Similarly, we use $m \circ x = x \circ m = mx + xm$ for all $x \in \mathcal{A}$ and

for all $m \in \mathcal{M}$. A linear map $f : \mathcal{A} \rightarrow \mathcal{M}$ is a generalized Jordan derivation, if there exists a linear map $d : \mathcal{A} \rightarrow \mathcal{M}$ such that

$$f(x \circ y) = f(x)y + f(y)x + xd(y) + yd(x) \quad \text{for all } x, y \in \mathcal{A}. \quad (4.1)$$

A linear map $f : \mathcal{A} \rightarrow \mathcal{M}$ such that $f(x \circ y) = f(x) \circ y + x \circ f(y)$ for all $x, y \in \mathcal{A}$ is called a Jordan derivation. Obviously, each Jordan derivation f is a generalized Jordan derivation, where $d = f$. Note that any generalized derivation is also a generalized Jordan derivation.

In 1957 Herstein [18] proved that every Jordan derivation from a prime ring of characteristic not 2 into itself is a derivation. This result has been extended to different rings and algebras in various directions (see e.g. [8, 10, 12, 23, 29] and references therein); one might very roughly summarize these results by saying that proper Jordan derivations (i.e. those that are not derivations) from rings (algebras) into themselves are rather rare and very special. Jing and Lu [24] have proven (with the assumption, that the associated map d is a Jordan derivation) that each generalized Jordan derivation of a prime algebra is a generalized derivation. Later, Vukman [32] extended this result to semiprime rings using a result on left Jordan centralizers [35].

We begin with the following basic result:

Lemma 4.1. *Let \mathcal{A} be an algebra with identity and let \mathcal{M} be a unital \mathcal{A} -bimodule. Let $f : \mathcal{A} \rightarrow \mathcal{M}$ be a generalized Jordan derivation with an associated linear map d . If \mathcal{M} is a 2-torsionfree \mathcal{A} -bimodule then d is a Jordan derivation and*

$$f(x) = f(1)x + d(x) \quad \text{for all } x \in \mathcal{A}. \quad (4.2)$$

Proof. Setting $x = y = 1$ in (4.1) we obtain $2f(1) = 2f(1) + 2d(1)$. Since \mathcal{M} is a 2-torsionfree \mathcal{A} -bimodule it follows $d(1) = 0$. Consequently, setting $y = 1$ in (4.1) we get $f(x) = f(1)x + d(x)$ for all $x \in \mathcal{A}$. It remains to prove that d is a Jordan derivation. Since $f(x) = f(1)x + d(x)$ (4.1) implies

$$\begin{aligned} f(1)(x \circ y) + d(x \circ y) &= f(1)xy + d(x)y + f(1)yx + d(y)x + xd(y) + yd(x) \\ &= f(1)(x \circ y) + d(x) \circ y + x \circ d(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. Therefore, d is a Jordan derivation. \square

Zhang and Yua [36, Theorem 2.1] have proven that any Jordan derivation d of a 2-torsionfree triangular algebra is a derivation. Using [36, Theorem 2.1] and Lemma 4.1 we obtain the following description of generalized Jordan derivations of a triangular algebra.

Theorem 4.2. *Let $\mathcal{A} = \text{Tri}(A, M, B)$ be a 2-torsionfree triangular algebra. Let $f : \mathcal{A} \rightarrow \mathcal{A}$ be a generalized Jordan derivation with an associated linear map d . Then f is a generalized derivation of the form $f(x) = f(1)x + d(x)$ for all $x \in \mathcal{A}$.*

In [19, Theorem 2.1] additive generalized Jordan derivations (whose associated maps are additive Jordan derivations) of a nest algebra are described. This result can be generalized and we can give a short and elegant proof. Although we assume in Corollary 4.3 that all maps are linear, it suffices to assume only additivity. Namely, Lemma 4.1 [36, Theorem 2.1] and Theorem 4.2 hold true also for additive maps.

Corollary 4.3. *Let \mathcal{N} be a nest of a Banach space X and let $\mathcal{T}(\mathcal{N})$ be a nest algebra. Let $f : \mathcal{T}(\mathcal{N}) \rightarrow \mathcal{T}(\mathcal{N})$ be a generalized Jordan derivation with an associated linear map d . If $\mathcal{N} = \{0, X\}$ or if there exists $N \in \mathcal{N} \setminus \{0, X\}$, which is complemented in X then f is a generalized derivation.*

Proof. If $\mathcal{N} = \{0, X\}$ then $\mathcal{T}(\mathcal{N}) = \mathcal{B}(X)$. Since $\mathcal{B}(X)$ is a prime algebra it follows that all Jordan derivations of $\mathcal{B}(X)$ are derivations (see [18]). Moreover, since $\mathcal{B}(X)$ contains identity Lemma 4.1 implies the desired result. If there exists $N \in \mathcal{N} \setminus \{0, X\}$ such that N is complemented in X then $\mathcal{T}(\mathcal{N})$ is a triangular algebra and the desired result follows from Theorem 4.2. \square

Let H be a Hilbert space and \mathcal{N} be a nest of H . If $\mathcal{N} = \{0, X\}$ then $\mathcal{T}(\mathcal{N}) = \mathcal{B}(H)$ is a prime algebra. If \mathcal{N} is a nontrivial nest then $\mathcal{T}(\mathcal{N})$ is a triangular algebra. Since all derivations of $\mathcal{T}(\mathcal{N})$ are inner we have the following result.

Corollary 4.4. *Let \mathcal{N} be a nest of a Hilbert space H and let $\mathcal{T}(\mathcal{N})$ be a nest algebra. Then any generalized Jordan derivation $f : \mathcal{T}(\mathcal{N}) \rightarrow \mathcal{T}(\mathcal{N})$ is a generalized inner derivation.*

Let R be a 2-torsionfree commutative ring with identity. Let $\mathcal{B}_n(R)$ be a block upper triangle matrix algebra and let $\mathcal{T}_n(R)$ be a upper triangular matrix algebra.

Corollary 4.5. *Any generalized Jordan derivation $f : \mathcal{B}_n(R) \rightarrow \mathcal{B}_n(R)$ is a generalized inner derivation. In particular, any generalized Jordan derivation of $\mathcal{T}_n(R)$ is a generalized inner derivation.*

Proof. Recall that all derivations of $\mathcal{B}_n(R)$ are inner. If $\mathcal{B}_n(R) \neq \mathcal{M}_n(R)$ then $\mathcal{B}_n(R)$ is a triangular algebra and the result follows from Theorem 4.2. Next, suppose that $\mathcal{B}_n(R) = \mathcal{M}_n(R)$. Using Lemma 4.1 and the fact that $\mathcal{M}_n(R)$ does not have proper Jordan derivations we obtain the conclusion. \square

Let $f : \mathcal{A} \rightarrow \mathcal{M}$ be a generalized Jordan derivation with an associated linear map d . Since \mathcal{A} has identity and \mathcal{M} is a 2-torsionfree bimodule then it follows from the proof of Lemma 4.1 that $d(x) = f(x) - f(1)x$ for all $x \in \mathcal{A}$. Consequently, (4.1) yields

$$f(x \circ y) = f(x) \circ y + x \circ f(y) - xf(1)y - yf(1)x \quad \text{for all } x, y \in \mathcal{A}. \quad (4.3)$$

Let us remark that in papers [28,37] a linear map $f : \mathcal{A} \rightarrow \mathcal{M}$ satisfying (4.3) is called a generalized Jordan derivation, while a linear map $f : \mathcal{A} \rightarrow \mathcal{M}$ satisfying

$$f(xy) = f(x)y + xf(y) - xf(1)y \quad \text{for all } x, y \in \mathcal{A} \quad (4.4)$$

is called a generalized derivation. Note that our definition of a generalized derivation is equivalent to the one given in (4.4) if \mathcal{A} is a unital algebra. Obviously, if \mathcal{M} is a 2-torsionfree bimodule then our definition of a generalized Jordan derivation is equivalent to the one given in (4.3). Now, recall that a linear map $\delta : \mathcal{A} \rightarrow \mathcal{M}$ such that $\delta(xy) = \delta(y)x + y\delta(x)$ for all $x, y \in \mathcal{A}$ is called an antiderivation. We end this paper with yet another corollary. Namely, our approach enables us to shorten the proof of the main result in [28, Theorem 2.4]:

Corollary 4.6. *Let \mathcal{M} be a 2-torsionfree $\mathcal{T}_n(R)$ -bimodule. Any generalized Jordan derivation $f : \mathcal{T}_n(R) \rightarrow \mathcal{M}$ is the sum of a generalized derivation $\Delta : \mathcal{T}_n(R) \rightarrow \mathcal{M}$ and an antiderivation $\delta : \mathcal{T}_n(R) \rightarrow \mathcal{M}$.*

Proof. Lemma 4.1 implies that $f(x) = f(1)x + d(x)$ for all $x \in \mathcal{A}$, where $d : \mathcal{T}_n(R) \rightarrow \mathcal{M}$ is a corresponding Jordan derivation. By [3, Theorem 1.1] we know that $d = d' + \delta$, where $d' : \mathcal{T}_n(R) \rightarrow \mathcal{M}$ is a derivation and $\delta : \mathcal{T}_n(R) \rightarrow \mathcal{M}$ is an antiderivation. Let $\Delta(x) = f(1)x + d'(x)$ for all $x \in \mathcal{A}$. Thus, Δ is a generalized derivation and $f = \Delta + \delta$. \square

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